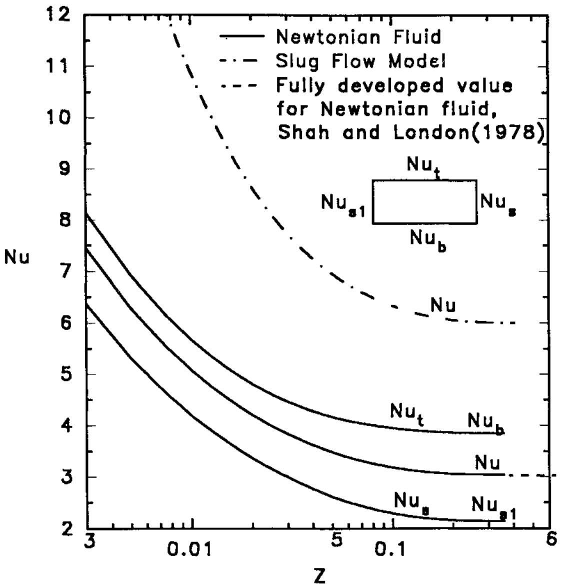

Fig. 2 Thermally developing Nusselt numbers as a function of *Z* for *H*2 boundary condition, with $\alpha^* = 1/2$.

Table 2 Comparison of thermal entrance lengths under various boundary conditions

α^*	Present L^*_{thH2}	Hartnett and Kostic (1989) L^*_{thH1}	L^*_{thT}
0.000	∞	0.0125	0.0080
0.100	1.312	0.0260	0.0422
0.125	0.883	0.0300	0.0450
0.167	0.550	0.0336	0.0500
0.200	0.403	0.0370	0.0521
0.250	0.290	0.0421	0.0541
0.333	0.175	0.0482	0.0539
0.400	0.135	0.0520	0.0520
0.500	0.104	0.0566	0.0490
0.750	0.077	0.0636	0.0446
1.000	0.065	0.0660	0.0410

identical, (i.e., $Nu_b = Nu_t$, $Nu_s = Nu_{s1}$). The corresponding slug flow model ($W = 1$) is included as a dash line in this figure. Although the slug flow solutions for various aspect ratios are not presented here, our computation reveals that the fully developed Nusselt number for slug flow approaches a value of 6.00, independent of the duct aspect ratio. This is in contrast to the case of Newtonian fluids for which the fully developed Nusselt number for square duct ($\alpha^* = 1$) appears to be the upper bound for all curves shown in Fig. 1.

The thermal entrance length which is defined as the duct length at which the local Nusselt number has reached 5% of the fully developed value, L^*_{th} is listed in Table 2 for some selected values of α^*. The comparison of L^*_{th} is made under aforementioned boundary conditions. Unlike the case of *T* and *H*1 boundary conditions reported by Hartnett and Kostic,[2] the thermal entrance length for *H*2 boundary condition does not approach the same value for plane parallel plate as $\alpha^* \to 0$. Instead, L^*_{thH2} goes to infinity when α^* approaches zero.

Conclusion

The forced convective heat transfer for the laminar flow under the *H*2 boundary condition in rectangular ducts is studied numerically. The developing Nusselt number is obtained for a wide range of duct aspect ratios. The limiting solution of Nu_{H2} as $Z \to \infty$ agrees excellently with those found in the literature. The thermal entrance length for this case shows a totally different trend from the counterpart of *T* and *H*1 boundary conditions. It increases monotonically as α^* decreases from one to zero.

The present analysis and discussion are restricted to the case of equal heating on adjacent walls. The analysis can be easily extended to the case of unequal heatings on adjacent walls. This latter case has significant applications in electronic industry. Some numerical examples for unequal heatings on adjacent walls for rectangular ducts were illustrated by these authors, see Chung et al.[8]

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Approximation Method for Rate of Appearance of Temperature Distributions in Spherical Objects

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Introduction

HEAT flow equations are the basis of many engineering applications, but the complexity of the equations makes it difficult to estimate temperature distributions, even in a semiquantitative manner. It would therefore be useful to have a method to approximate the rate at which the temperature distributions develop during the course of heat treatment.

Model for Spherical Objects

For the purposes of this model, the object will be assumed to have spherical symmetry and uniform composition, where

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the radial distance is r and the outer radius of the sphere is a. It is initially at a uniform ambient temperature T_a, and at time t equal to zero, is placed in an infinite heat bath at temperature T_b.

The heat flow equations for a sphere[1] yield the equation for the temperature $T(r, t)$ at r and t

$$T(r, t) = T_a + [(T_b - T_a)]x \left\{ 1 + (2a/\pi r) \cdot \sum_{n=1}^{\infty} [(-1)^n/n][\sin(n\pi r/a)]\exp(-\kappa n^2\pi^2 t/a^2) \right\} \tag{1}$$

where κ is the thermal diffusivity, equal to K/C_p, K being the thermal conductivity and C_p the volumetric heat capacity.

Half-Heating Time Approximation

A property that has been profitably used in cooling-time studies is the half-cooling time,[2] which is the time at which the temperature difference between the object and its surroundings is one-half the initial temperature difference. By analogy, we define the "half-heating time" $\tau_{1/2}$ to be the time at which the temperature difference between the object and its original ambient temperature is one-half of the initial temperature difference, $T_b - T_a$. That is, at $\tau_{1/2}$

$$T(r, \tau_{1/2}) - T_a = 0.5(T_b - T_a)$$

Defining the reduced radius σ as r/a, the half-heating condition yields

$$-\pi\sigma/4 = \sum_{n=1}^{\infty} [(-1)^n/n][\sin(n\pi\sigma)]\exp(-\kappa n^2\pi^2\tau_{1/2}/a^2) \tag{2}$$

The first approximation is to truncate the sum after one term

$$\pi\sigma/4 \cong \sin(\pi\sigma)\exp(-\kappa\pi^2\tau_{1/2}/a^2) \tag{3}$$

$$4\sin(\pi\sigma)/\pi\sigma \cong \exp(\kappa\pi^2\tau_{1/2}/a^2) \tag{4}$$

Taking the natural logarithm of Eq. (4)

$$\ln 4 - \ln(\pi\sigma) + \ln[\sin(\pi\sigma)] \cong \kappa\pi^2\tau_{1/2}/a^2 \tag{5}$$

The second approximation is to expand $\ln[\sin(\pi\sigma)]$ as a series[3]

$$\ln(\sin x) \cong \ln x - x^2/6 - x^4/180 - x^6/2835 + \ldots \tag{6}$$

and to truncate after the second term

$$\ln[\sin(\pi\sigma)] \cong \ln(\pi\sigma) - (\pi\sigma)^2/6 \tag{7}$$

Now Eq. (5) reduces to

$$\ln 4 - (\pi\sigma)^2/6 = \kappa\pi^2\tau_{1/2}/a^2 \tag{8}$$

Solving for the half-heating time

$$\tau_{1/2} \cong (a^2/\kappa\pi^2)[\ln 4 - (\pi\sigma)^2/6] \tag{9}$$

Testing the Half-Heating Time Equation

We will compare the results of the predictions of Eq. (9) with the exact numerical results from computer simulation for two models whose parameters are: $a = 1.0$ cm, $T_a = 300$ K, $T_b = 1000$ K and $\kappa = 0.010$ cm^2/h (for model 1) and 0.0050 cm^2/h (for model 2). The computational method used to solve Eq. (1) numerically has been previously described.[4]

If Eq. (9) is valid, a plot of the half-heating time vs the square of σ should be linear with slope equal to $-a^2/6\kappa$ and intercept equal to $a^2 \ln 4/\kappa\pi^2$. Figure 1 displays a graph of exact half-heating times from a computer simulation for these models vs σ^2 over the range $0.1 \leq \sigma \leq 0.9$. The graphs are clearly linear, with correlation coefficients R equal to 0.998 in both cases. The slopes are equal to -16.66 h/cm^2 and

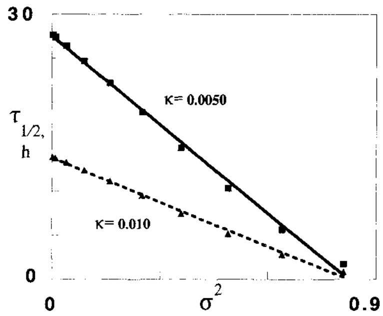

Fig. 1 $\tau_{1/2}(h)$ from exact computation vs σ^2 for models 1 and 2 (κ = 0.010 and 0.0050 cm^2/h). The lines are the best linear fits.

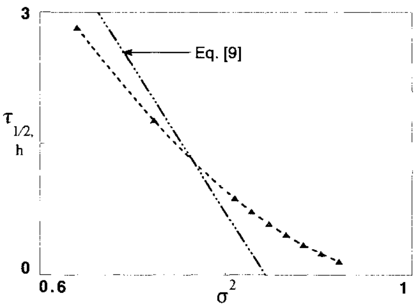

Fig. 2 $\tau_{1/2}(h)$ from exact computation as data points (with an interpolated line) vs σ^2 for model 1 for σ between 0.8 and 0.96. The prediction of Eq. (9) is displayed.

-33.31 h/cm^2, almost exactly the values predicted from Eq. (9): -16.67 h/cm^2 and -33.33 h/cm^2. The intercepts are 13.75 and 27.50 h, while the predicted values are 14.05 and 28.10 h. Clearly, Eq. (9) is valid over most of the range of σ.

It is necessary, however, to determine those conditions that will render this equation invalid. This should occur when either of the two approximations fail, that is, when conditions, Eqs. (10) and (11) below, do not hold

$$(\pi\sigma)^4/180 << (\pi\sigma)^2/6 \tag{10}$$

$$-(\tfrac{1}{2})\sin(2\pi\sigma)\exp(-4\kappa\pi^2\tau_{1/2}/a^2) << \sin(\pi\sigma)\exp(-\kappa\pi^2\tau_{1/2}/a^2) \tag{11}$$

Condition Eq. (10) results from the truncation of Eq. (6), and clearly implies that the approximation should fail at large values of σ. Moreover, the expansion, Eq. (6), of $\ln\sin(x)$ is only valid in the range $(0 < x < \pi)$. The factors that lead condition Eq. (11) to fail, are more complex. One factor will arise if $\exp(-4\kappa\pi^2\tau_{1/2}/a^2)$ is not much smaller than $\exp(-\kappa\pi^2\tau_{1/2}/a^2)$, leading to the expectation that the approximation will fail at the small half-heating times experienced close to the periphery. Thus, the critical region to test these approximations is for σ near 1.

Figure 2 displays the behavior of the exact half-heating time as a function of σ^2 in the range $0.8 < \sigma < 0.96$, showing that the predicted behavior is not observed in this region. Testing conditions Eqs. (10) and (11) for $\sigma = 0.9$, it is found that both fail. Small values of $\tau_{1/2}$ cannot be well-represented by Eq. (9), which gives negative values at $\sigma > 0.92$.

Discussion

The approximate equation for $\tau_{1/2}$ shows that it depends on the radial distance r in a quadratic manner

$$\tau_{1/2} = \alpha r^2 + \beta \tag{12}$$

where

$$\alpha = -\tfrac{1}{6}\kappa \tag{13}$$

$$\beta = a^2 \ln 4/\pi^2\kappa \tag{14}$$

Thus, for material of given composition and fixed κ, $\tau_{1/2}$ will change with the square of the radial distance within the spherical object, with a sensitivity inversely proportional to the thermal diffusivity. For a given value of r, $\tau_{1/2}$ will depend only on κ and a, again in an expected manner: for large a the half-heating times become large, also in a quadratic manner.

A quick qualitative sketch of the thermal gradients can be made if one recognizes that $\tau_{1/4}$ and $\tau_{3/4}$ are given by

$$\tau_{1/4} = (a^2/\kappa\pi^2)\{\ln(8/3) - [(\pi\sigma)^2/6]\} \tag{15}$$

$$\tau_{3/4} = (a^2/\kappa\pi^2)\{\ln 8 - [(\pi\sigma)^2/6]\} \tag{16}$$

In conclusion, the concept of the half-heating time offers a simple manner of estimating the rate of appearance of temperature distributions and the manner in which they depend on the physical properties of the system. For spherical objects, it is approximately given by a remarkably simply equation whose accuracy extends over a wide range of conditions.

Acknowledgments

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Analytical Solution for Thermal Runaway in the Surface Heating of Plates

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Nomenclature

$C(\lambda)$ = defined in Eqs. (13) and (15)
c = plate thickness or half thickness, Eqs. (3) and (10)
h = convective coefficient
k = thermal conductivity
q = absorbed incident surface heat flux
q_i = incident surface heat flux
T = temperature
t = time
x, y, z = spatial coordinates
α = constant, Eq. (1)
β = real constant defined in Eq. (8)
κ = thermal diffusivity
λ = indexed eigenvalues
ρ = surface absorptance
ρ_0 = constant surface absorptance
$\phi(x)$ = eigenfunction

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Introduction

THERE are many instances in high energy systems in which the input heat flux to a surface is dependent on the temperature of the surface. One example is the laser heating of a metal surface in which the surface absorptance increases linearly with surface temperature.[1,2] Another example is linear particle accelerators in which the resistive losses in the cavity are proportional to the cavity surface temperature. The temperature solutions to such problems can be unstable in the sense that the surface temperature increases exponentially with time. In many cases this situation is undesirable and the rear surface (the surface opposite the heating) is cooled convectively. An analytical solution will be developed for the case of an insulated rear surface and for a convectively cooled rear surface. The stability of these temperature solutions will be examined.

Problem

A simple problem to illustrate the solution technique will now be described. A laser irradiates a metal plate that extends far into the y and z directions so that all the temperature variation is in the x direction, i.e., the problem is one-dimensional. The plate is insulated on the rear surface (an excellent approximation for short time high energy laser heating of a thin plate) as shown in Fig. 1.

The front surface absorptance is given by:

$$\rho = \rho_0 + \alpha T \tag{1}$$

The fact that the absorptance is given, to an excellent approximation, by Eq. (1) is well documented.[2] The heat flux is then given by

$$q = \rho q_i = \rho_0 q_i + \alpha q_i T \tag{2}$$

The governing equation and boundary conditions are then given by

$$\begin{aligned} &\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \\ &T(x, 0) = 0 \quad \text{for} \quad 0 < x < c \\ &-k\frac{\partial T}{\partial x} = \rho q_i = \rho_0 q_i + \alpha q_i T \quad @ \quad x = 0 \\ &-k\frac{\partial T}{\partial x} = 0 \quad @ \quad x = c \end{aligned} \tag{3}$$

The solution to Eq. (3), using the method described in the Appendix is

$$T(x, t) = -\frac{\rho_0}{\alpha} - \frac{\rho_0 q_i}{k}\sum_{m=0}^{\infty}\frac{C(\lambda_m)\phi_m(x)\cos(\lambda_m c)}{\lambda_m^2} \cdot \exp(-\kappa\lambda_m^2 t) \tag{4}$$

Where the various terms are defined in the Appendix. The eigenfunction is

$$\phi_m = \cos\lambda_m(x - c) \tag{5}$$

where λ_m is a root of

$$\lambda_m c\tan(\lambda_m c) = -(c\alpha q_i/k) \tag{6}$$